

Q-DIFFERENTIAL OPERATORS

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ABSTRACT. We set up a framework for discussing “ q -analogues” of the usual covariant differential operators for hermitian symmetric spaces. This turns out to be directly related to the deformation quantization associated to quadratic algebras satisfying certain conditions introduced by Procesi and De Concini.

1. INTRODUCTION

The investigation, of which we are here reporting some results, began with the question about what should be “quantized wave operators” in the context of (quantized) hermitian symmetric spaces. Immediately, there is a very simple thing one can do, namely one can pass to the quantized enveloping algebra. Here, there are unitarizable highest weight modules and the most singular of these have kernels which, in analogy with the case $q = 1$ can be said to be “quantized wave operators” ([Dob95], [Jak97]). However, when q is generic, there is no immediate space of functions on which these differential operators act.

The first objects we have come across in our attempt to repair on this are (families of) quadratic algebras that seem to replace the hermitian symmetric spaces. See [JJJZ98], [Jak96], and below. Secondly, a natural setting for differential operators (in an algebraic approach) could be duality. Combining these two one comes across the following:

Let \mathcal{P} be a projection (not necessarily self adjoint) in the tensor algebra $T(V)$ over some (finite-dimensional) vector space. Suppose that \mathcal{P} maps $T^r(V)$ to $T^r(V)$ for each r . Solutions \mathcal{P} of (\star) or $(\star\star)$ to the following equations, reminiscent of the Yang-Baxter equations, turn out to have a fundamental importance.

$$\begin{array}{ll} (\star) & \forall r, s : (I_r \otimes \mathcal{P} \otimes I_s) \mathcal{P} = \mathcal{P} \\ (\star\star) & \forall r, s : \mathcal{P}(I_r \otimes \mathcal{P} \otimes I_s) = \mathcal{P} \end{array}$$

Indeed, such a partial solution can be used to define an associative algebra of polynomial functions on either V (case of (\star)) or V^* (case of $(\star\star)$). And, once this has been established, one may introduce, by duality, quantized differential operators.

We will construct below, for a quadratic algebra that satisfies a certain technical condition, a projection \mathcal{P} (a quantized symmetrization map) which solves both equations at the same time. The condition is related to the condition in “the Diamond Lemma” by Bergman ([Ber78]) – a major influence for us in relation to this part. The condition is satisfied by the quadratic algebras connected with hermitian symmetric spaces.

One aspect of some of the quadratic algebras that fulfill the condition (including those from hermitian symmetric spaces) is that they give rise to Poisson structures. The deformed products obtained from \mathcal{P} is directly related to this in the usual way.

Our way of quantizing holomorphic functions may be extended to all functions by quantizing anti-holomorphic functions independently and then, based on considerations involving e.g. reproducing kernels, letting holomorphic and anti-holomorphic variables commute. We mention that other possibilities have been extensively studied by, in particular, D. Shklyarov, S. Sinel'shchikov, and L. Vaksman. See e.g. [VS97] and references cited therein, or math.QA. at <http://xxx.lanl.gov/>.

The material is organized as follows: In Section 2 we give a short description of the way covariant differential operators arise in the classical case, c.f. [HJ83], [Jak85]. In Section 3 and Section 4, quadratic algebras are introduced, examples are given, and some technical assumptions are discussed. Then, in Section 5 the operators \mathcal{P} are finally constructed and basic properties are given. The associative (non-commutative) polynomial algebras are introduced via duality in Section 6, and it is briefly discussed how different choices of bases may give different presentations of the same algebra. The quantized differential operators are then introduced by means of the duality. In Section 7 the situation is analyzed in detail for $M_q(2)$. The “ q -differential operators” are seen to consist of some rather agreeable components together with possibly a more complicated term which points towards covariant differentiation in infinite dimensional spaces. Further aspects of this will be presented in forthcoming papers. Finally, some computations of the differential operators for $M_q(n)$ are appended.

2. THE CLASSICAL SITUATION OR HOW TO GET THE WAVE OPERATOR, THE DIRAC OPERATOR, MAXWELL'S EQUATIONS ETC. (IN THE MASS 0 CASE/ABSENCE OF SOURCES CASE) WITHOUT PHYSICS.

Let \mathcal{B} be an irreducible hermitian symmetric space of the noncompact type. Then (c.f. Helgason [Hel62, Chapter VIII]) \mathcal{B} is diffeomorphic to G/K where G is a connected noncompact simple Lie group with trivial center and K is a maximal compact subgroup with non-discrete center. If $\mathfrak{g}, \mathfrak{k}$ denote the complexified Lie algebras of G, K , respectively, then there are complex subalgebras \mathfrak{p}^\pm such that

$$\begin{aligned} \mathfrak{g} &= \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+, \\ [\mathfrak{p}^\pm, \mathfrak{p}^\pm] &= 0, \\ [\mathfrak{p}^+, \mathfrak{p}^-] &\subseteq \mathfrak{k}, \quad \text{and} \quad [\mathfrak{k}, \mathfrak{p}^\pm] \subseteq \mathfrak{p}^\pm. \end{aligned} \tag{1}$$

Moreover, we choose a subalgebra \mathfrak{h} which is both a Cartan subalgebra for \mathfrak{g} and \mathfrak{k} . Observe that we have

$$\boxed{\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{p}^-) \cdot \mathcal{U}(\mathfrak{k}) \cdot \mathcal{U}(\mathfrak{p}^+).} \tag{2}$$

Operationally, it is here more adequate to use the equivalent description where \mathcal{B} is a bounded symmetric domain in \mathbb{C}^N , G is the connected component of the group of biholomorphic bijections of \mathcal{B} onto itself, and K is the isotropy group of a point. Indeed, we may, and shall, take \mathcal{B} to be an open bounded subset of \mathfrak{p}^- such that $0 \in \mathcal{B}$ and such that K acts linearly.

Let τ be a unitary representation of K in a finite dimensional vector space V_τ . Then $G \times_K V_\tau$ is a vector bundle over \mathcal{B} and G acts naturally on the space $\Gamma_h(G \times_K V_\tau)$ of holomorphic sections of $G \times_K V_\tau$. The bundle is equivalent to a trivial bundle $\mathcal{B} \times V_\tau$ and as a result one obtains a representation U_τ of G in the space $\mathcal{H}(\mathfrak{p}^-) \otimes V_\tau$ of V_τ valued holomorphic

functions on \mathcal{B} . The algebraic span of the K types is exactly the space $\mathcal{P}(\mathfrak{p}^-) \otimes V_\tau$ of V_τ valued polynomials on \mathfrak{p}^- .

We say that a differential operator

$$\mathcal{D} : \mathcal{H}(\mathfrak{p}^-) \otimes V_{\tau_1} \mapsto \mathcal{H}(\mathfrak{p}^-) \otimes V_{\tau_2}$$

is covariant provided

$$\forall g \in G : U_{\tau_2}(g)\mathcal{D} = \mathcal{D}U_{\tau_1}(g). \quad (3)$$

It follows from the assumptions that \mathcal{D} is a holomorphic constant coefficient $\text{hom}(V_{\tau_1}, V_{\tau_2})$ valued differential operator.

It turns out that such operators indeed do exist, even under additional unitarity assumptions, but to get a better understanding of where they come from, we turn to another construction:

Definition 2.1. *For V_τ as before,*

$$M(V_\tau) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k} + \mathfrak{p}^+)} V_\tau$$

is called a generalized Verma module. It is a highest weight module generated by a non-zero highest weight vector v_τ . Specifically, $\mathfrak{p}^+ v_\tau = \mathfrak{k}^+ v_\tau = 0$ and $\forall h \in \mathfrak{h} : h \cdot v_\tau = \Lambda_\tau(h) \cdot v_\tau$ for some (highest weight) $\Lambda_\tau \in \mathfrak{h}^$.*

The analogue of a covariant operator at this level is a $\mathcal{U}(\mathfrak{g})$ homomorphism $\phi : M(V_{\tau_2}) \mapsto M(V_{\tau_1})$. A homomorphism ϕ is completely determined by $\hat{v}_{\tau_2} = \phi(1)$ - a vector in $M(V_{\tau_1})$ which has the same weight as v_{τ_2} and which is annihilated by \mathfrak{p}^+ and \mathfrak{k}^+ . Conversely any such so called primitive vector (for physicists: a secondary vacuum) determines a homomorphism.

The key fact now is the following

Proposition 2.2. *There is a natural non-degenerate pairing between $\mathcal{P} \otimes V_\tau$ and $M(V_{\tau'}) = M(V_{\tau'})$ under which the spaces as $\mathcal{U}(\mathfrak{g})$ modules are the dual to each other. Under this duality, a homomorphism between generalized Verma modules correspond to a covariant differential operator in the dual picture - and conversely, a covariant differential operator determines in the dual picture a homomorphism.*

Another key fact is that there occur naturally some homomorphisms between generalized Verma modules at singular unitary holomorphic representations. Indeed, the homomorphism is defined in terms of the lowest “missing \mathfrak{k} type”.

Consider the symmetric algebras

$$\boxed{S(\mathfrak{p}^\pm) = T(\mathfrak{p}^\pm) / I_\pm(XY - YX)} \quad (4)$$

where $I_\pm(XY - YX)$ denotes the ideal in $T(\mathfrak{p}^\pm)$ generated by all elements of the form $X \otimes Y - Y \otimes X$ with $X, Y \in \mathfrak{p}^\pm$. This is clearly a quadratic algebra. Let \mathcal{P}_0^\pm denote the projections of $T(\mathfrak{p}^\pm)$ onto $S(\mathfrak{p}^\pm)$. These are well known maps:

$$\boxed{\mathcal{P}_0^\pm \text{ are symmetrization maps.}} \quad (5)$$

The Killing form B on \mathfrak{g} gives a non-degenerate pairing between \mathfrak{p}^+ and \mathfrak{p}^- . For $w^+ \in \mathfrak{p}^+$ and $z^- \in \mathfrak{p}^-$ we write $\langle w^+, z^- \rangle = B(w^+, z^-)$. This extends to a pairing between $S(\mathfrak{p}^+)$ and $S(\mathfrak{p}^-)$ by

$$\begin{aligned} \langle [w_1^+ \otimes \cdots \otimes w_r^+], [z_1^- \otimes \cdots \otimes z_s^-] \rangle &= \delta_{r,s} \sum_{\sigma \in S_r} \prod_{i=1}^r \langle w_i^+, z_{\sigma(i)}^- \rangle \\ &= \delta_{r,s} r! \langle w_1^+ \otimes \cdots \otimes w_r^+, \mathcal{P}_0^-(z_1^- \otimes \cdots \otimes z_s^-) \rangle. \end{aligned} \quad (6)$$

Through this pairing, any element $[w] = [w_1^+ \otimes \cdots \otimes w_r^+] \in S(\mathfrak{p}^+)$ defines a polynomial $\mathcal{F}_{[w]}^0 \in \mathcal{P}(\mathfrak{p}^-)$ by

$$\boxed{\mathcal{F}_{[w]}^0(z^-) = \langle \mathcal{P}_0^+(w_1^+ \otimes \cdots \otimes w_r^+), z^- \otimes \cdots \otimes z^- \otimes \cdots \rangle.} \quad (7)$$

In this way we get an identification of vector spaces (indeed, of \mathfrak{k} modules)

$$\mathcal{P}(\mathfrak{p}^-) \otimes V_\tau = S(\mathfrak{p}^+) \otimes V_\tau. \quad (8)$$

Similarly,

$$M(V'_\tau) = S(\mathfrak{p}^-) \otimes V'_\tau, \quad (9)$$

and the pairing between the two modules is just the introduced pairing between $S(\mathfrak{p}^+)$ and $S(\mathfrak{p}^-)$ augmented with the pairing between the module V_τ and its dual module V'_τ .

Now observe that \mathfrak{p}^- acts on $\mathcal{P}(\mathfrak{p}^-)$ by contraction,

$$\boxed{\begin{aligned} (z_0^- \mathcal{F}_{[w]}^0)((z^-)) &= \langle (w_1^+ \otimes \cdots \otimes w_r^+), \mathcal{P}_0^-(z_0^- \otimes z^- \otimes \cdots \otimes z^- \otimes \cdots) \rangle \\ &= \left(\frac{\partial}{\partial z_0^-} \mathcal{F}_{[w]}^0 \right)((z^-)). \end{aligned}} \quad (10)$$

But this is just a differentiation, and in this way, $S(\mathfrak{p}^-)$ can be viewed as either a space of polynomials on \mathfrak{p}^+ or as a space of constant coefficient differential operators on \mathfrak{p}^- . Extending the above to the case of generalized Verma modules, $M(V'_\tau)$ is the space of V'_τ valued constant coefficient differential operators on \mathfrak{p}^- and the pairing above can be formulated as follows: If $z^- \mapsto p^-(z^-) \otimes v \in \mathcal{P}(\mathfrak{p}^-) \otimes V_\tau$ and if $p^+(\frac{\partial}{\partial z^-}) \otimes v' \in M(V'_\tau)$ then

$$\langle p^- \otimes v, p^+ \otimes v' \rangle = \left(p^+ \left(\frac{\partial}{\partial z^-} \right) p^- \right) (0) \cdot \langle v, v' \rangle. \quad (11)$$

Finally, observe that the product in e.g. $\mathcal{P}(\mathfrak{p}^-)$ is given by

$$\boxed{\mathcal{F}_{[w^a]}^0 \star \mathcal{F}_{[w^b]}^0 = \mathcal{F}_{[w^a \otimes w^b]}^0.} \quad (12)$$

The well-definedness of this follows from

$$\boxed{(I_r \otimes \mathcal{P}_0^+ \otimes I_s) \mathcal{P}_0^+ = \mathcal{P}_0^+(I_r \otimes \mathcal{P}_0^+ \otimes I_s) = \mathcal{P}_0^+.} \quad (13)$$

This star is commutative simply because we work with real symmetrization.

3. QUADRATIC ALGEBRAS

Our construction below, though inspired by hermitian symmetric spaces, works for a more general class of algebras, namely quadratic algebras (subject to some technical assumptions to be stated later). We first give some examples and then later the precise definitions.

3.1. Examples of quadratic algebras. The simplest quadratic algebras are the commutative ones

$$X_i X_j - X_j X_i = 0.$$

We see that the symmetric algebras of the previous section fall in this category. Other examples are “quantized objects” e.g.

$$\begin{aligned} AB &= qBA && \text{(quantum plane)} \\ AB - q^2 BA &= 1 && \text{(quantized Weyl)}, \end{aligned}$$

where $q \in \mathbb{C}^*$ is the quantum parameter.

One of the most studied ones is the quantized function algebra of $n \times n$ matrices, $M_q(n)$, defined by the relations

$$\begin{aligned} \mathbf{AIII} : \quad Z_{i,j} Z_{i,k} &= q Z_{i,k} Z_{i,j} \text{ if } j < k, \\ Z_{i,j} Z_{k,j} &= q Z_{k,j} Z_{i,j} \text{ if } i < k, \\ Z_{i,j} Z_{s,t} &= Z_{s,t} Z_{i,j} \text{ if } i < s \text{ and } t < j, \\ Z_{i,j} Z_{s,t} &= Z_{s,t} Z_{i,j} + (q - q^{-1}) Z_{i,t} Z_{s,j} \text{ if } i < s \text{ and } j < t. \end{aligned} \tag{14}$$

This algebra is in the class of quadratic algebras connected with quantized hermitian symmetric spaces and for this reason we sometimes refer to it as **AIII**. We mention two more from the class, namely **CI** (but **DIII** is also covered by this) and **BDI** ($q = 2$). Observe that a misprint in the relations for **CI** has been corrected and two missing relations have been added compared to ([Jak96])

CI :

$$\begin{aligned} W_{i,i} W_{j,j} - W_{j,j} W_{i,i} &= \frac{1 - q^2}{q + q^{-1}} W_{i,j}^2 \text{ (} i < j \text{)}, \\ W_{i,i} W_{j,k} - W_{j,k} W_{i,i} &= (1 - q^2) W_{i,j} W_{i,k} \text{ (} i < j \text{ and } j < k \text{)}, \\ q W_{i,j} W_{j,k} - W_{j,k} W_{i,j} &= (q^{-2} - q^2) W_{i,k} W_{j,j} \text{ (} i < j < k \text{)}, \\ W_{i,j} W_{k,l} - W_{k,l} W_{i,j} &= q^{-1} W_{i,k} W_{j,l} - q W_{i,k} W_{j,l} \text{ (} i < j < k < l \text{)}, \\ W_{i,j} W_{k,k} - W_{k,k} W_{i,j} &= (1 - q^2) W_{i,k} W_{j,k}, \\ W_{i,j} W_{k,l} - W_{k,l} W_{i,j} &= (q^{-1} - q) W_{i,l} W_{k,j} \text{ (} i, k < j < l \text{)}, \\ W_{i,i} W_{i,j} &= q^{-2} W_{i,j} W_{i,i} \text{ (} i < j \text{)}, \\ W_{i,j} W_{j,j} &= q^{-2} W_{j,j} W_{i,j} \text{ (} i < j \text{)}, \\ W_{i,j} W_{i,k} &= q^{-1} W_{i,k} W_{i,j} \text{ (} i < j < k \text{)}, \\ W_{i,k} W_{j,k} &= q^{-1} W_{j,k} W_{i,k} \text{ (} i < j < k \text{)}, \\ W_{i,j} W_{k,l} &= W_{k,l} W_{i,j} \text{ (} i \leq j, k < i, \text{ and } j < l \text{)}. \end{aligned}$$

The relations for type **BDI** are:

$$\mathbf{BDI} : \quad W_i W_{i+r+1} = q^{-1} W_{i+r+1} W_i \text{ if } r \geq 0 \text{ and } r \neq 2(n-i), \tag{15}$$

$$W_i W_{2n+1-i} - W_{2n+1-i} W_i = -q W_{i+1} W_{2n-i} + q^{-1} W_{2n-i} W_{i+1} \tag{16}$$

$$\text{for } i = 1, \dots, n-1. \tag{17}$$

If we let $Z_i - 1 = (-q)^{-i} W_i$ for $i = 1, \dots, n$ and $Z_i^* = W_{2n-i}$ for $i = 0, \dots, n-1$ the last relations are seen (replacing n by N) to be those of the quantized Heisenberg space

(by some called the quantum symplectic space - a name which according to our classes is somewhat confusing), $F_q(N)$ of the quantum space \mathbb{C}^N , i.e. the associative algebra generated by $z_0, z_1, \dots, z_{N-1}, z_0^*, z_1^*, \dots, z_{N-1}^*$ subject to the following relations:

$$\begin{aligned} z_i z_j &= q^{-1} z_j z_i \text{ for } i < j, \\ z_i^* z_j^* &= q z_j^* z_i^* \text{ for } i < j, \\ z_i z_j^* &= q^{-1} z_j^* z_i \text{ for } i \neq j, \text{ and} \\ z_i z_i^* - z_i^* z_i &= (q^2 - 1) \sum_{k>i} z_k z_k^*. \end{aligned} \tag{18}$$

3.2. General definition. The general definition of a quadratic algebra is as follows:

Let V denote an N -dimensional complex vector space, let $T = T(V)$ denote the tensor algebra over V , let R be a subspace of $V \otimes V$, and let I_R denote the 2-sided ideal in T generated by the R . Then

Definition 3.1.

$$\mathcal{A} = T/I_R.$$

We say that I_R is the space generated by the relations.

The starting point of our present investigation is the following fact:

Let $\mathfrak{g}, \mathfrak{k}$ be as in the Section 2. Let $\mathcal{U}_q(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{k})$ be the quantized enveloping algebras of \mathfrak{g} and \mathfrak{k} , respectively. Then there are quadratic algebras \mathcal{A}^\pm which furthermore are $\mathcal{U}_q(\mathfrak{k})$ modules such that

$$\boxed{\mathcal{U}_q(\mathfrak{g}) = \mathcal{A}^- \cdot \mathcal{U}_q(\mathfrak{k}) \cdot \mathcal{A}^+}. \tag{19}$$

The quadratic algebras satisfy the additional assumptions below.

4. TECHNICAL DISCUSSION

We consider a quadratic algebra \mathcal{A} generated by (linearly independent) elements X_1, \dots, X_N . For each $i = 1, \dots, N$ let \mathcal{A}_i denote the algebra generated by X_1, \dots, X_i . We assume that the defining relations are of the form:

$$(\text{Rel}) \quad \text{If } i > j \text{ then } X_i X_j = b_{ij} X_j X_i + p_{ij}, \text{ with } p_{ij} \in \mathcal{A}_{i-1}.$$

Let V denote the N -dimensional complex vector space spanned by the elements X_1, \dots, X_N , let $T = T(V)$ denote the tensor algebra over V , and let I_R denote the ideal in T generated by elements $X_i X_j - (b_{ij} X_j X_i + p_{ij})$. Then

$$\mathcal{A} := T/I_R. \tag{20}$$

For $r \in \mathbb{N}$ we let $T^r = \underbrace{V \otimes \dots \otimes V}_r$. To an element $X = X_{i_1} \otimes X_{i_2} \otimes \dots \otimes X_{i_r} \in T^r$ we associate the element $\ell(X) = (n_1, n_2, \dots, n_N) \in \{0, 1, \dots, r\}^N$ where

$$\forall i = 1, \dots, N : n_i = \#\{s \mid i_s = i\}. \tag{21}$$

We shall from now on drop the \otimes whenever this can be done without placing the presentation in jeopardy.

We now introduce lexicographic ordering \leq_l on $\{0, 1, \dots, r\}^N$ (according to which $(0, \dots, 0, r)$ is the biggest element) to introduce a partial ordering, also denoted \leq_l , on the set of monomials in T^r simply by declaring $u' \leq_l u \Leftrightarrow \ell(u') \leq \ell(u)$.

The essential assumption (EA) which we now make is introduced to avoid situations where, due to some special cancelations, a sum of elements in I_R might add up to an element which strictly precedes all the summands in the order. Specifically, we assume for any element $u \in I_R$

(EA):

$$u \in \text{Span}\{a \cdot (X_i X_j - b_{ij} X_j X_i - p_{ij}) \cdot b \mid a, b \in T \text{ and } a \cdot (X_i X_j) \cdot b \leq_l u\}.$$

In the following we shall introduce certain operations which are related to thinking of (Rel) as a reduction system. The reductions are then of the form

$$(X_i X_j, b_{ij} X_j X_i + p_{ij}) \quad (\text{for all } i > j). \quad (22)$$

Indeed, we can, analogously to [Ber78, Section 3], introduce a *misordering index* $i(Z)$ of an element $Z = X_{i_1} \dots X_{i_r}$ as the number of pairs of indices (i_a, i_b) in Z for which $i_a > i_b$. This we can combine with the ordering \leq_l to give a new partial ordering, \leq on monomials as follows:

$$u_1 < u_2 \stackrel{\text{Def.}}{\Leftrightarrow} \begin{cases} u_1 <_l u_2 \text{ or} \\ \ell(u_1) = \ell(u_2) \text{ and } i(u_1) < i(u_2). \end{cases} \quad (23)$$

It is clear that if A, B are monomials in T , then $u' < u \Rightarrow A \cdot u \cdot B < A \cdot u' \cdot B$. Thus, our partial ordering is a *semigroup partial ordering*. Moreover, for all r, s with $s > r$ we have that $b_{sr} X_r X_s$ is of strictly smaller misordering index and p_{sr} is of strictly smaller lexicographic order than $X_s X_r$. Thus $b_{sr} X_r X_s + p_{sr}$ is of strictly less order (w.r.t $<$) than $X_s X_r$ and hence, the reduction system is *compatible with the reduction system*.

The two mentioned properties are parts of the requirements for the Diamond Lemma [Ber78, Theorem 1.2] to be applicable to our situation.

Proposition 4.1. *All elements of T are reduction unique.*

Proof: By observing that all reductions decrease the order it follows that the system satisfies the descending chain condition. It remains, according to [Ber78, p. 181], to prove that all ambiguities of the reduction system are resolvable. The only place where we can get ambiguities are on terms $X_i X_j X_k$ with $i > j > k$. Here we must prove (still following [Ber78, p. 181])

$$Y = (b_{ij} X_j X_i + p_{ij}) X_k - X_i (b_{jk} X_k X_j + p_{jk}) \in I_{i,j,k}, \quad (24)$$

where $I_{i,j,k}$ denotes the subspace spanned by all elements $A((X_s X_r - b_{sr} X_r X_s - p_{sr})B$ with $s > r$ and $A(X_s X_r)B < X_i X_j X_k$. But clearly, $(b_{ij} X_j X_i + p_{ij}) X_k - X_i (b_{jk} X_k X_j + p_{jk}) \in I_R$ (it is the reduction of $X_i X_j X_k - X_i X_j X_k$). Secondly, the only monomials in Y that map to $\ell(X_i X_j X_k)$ under the map ℓ are $b_{ij} X_j X_i X_k$ and $b_{jk} X_i X_k X_j$. But after two more reductions, they both become $b_{ij} b_{jk} b_{ik} X_k X_j X_i$ plus something of lower lexicographic order. Since the two original terms have opposite signs, the highest order terms cancel. The claim then follows from (EA). According to the Diamond Lemma we are done. \square

We immediately get

Corollary 4.2. *The set $\{X_1^{i_1} \cdots X_N^{i_N} \mid 1, \dots, i_n \in \mathbb{N}_0\}$ is a basis for \mathcal{A} .*

Corollary 4.3. *\mathcal{A} is a domain and is in fact an iterated twisted polynomial algebra. In particular, the assumptions of Procesi and De Concini ([DCP93]) are satisfied.*

Conversely we have the following result which implies that the algebras **AIII**, **BDI**, and **CI** above fit into the framework:

Proposition 4.4. *Given a quadratic algebra \mathcal{A} as above, satisfying (Rel), and furthermore satisfying*

(DCP) In all cases where $j < i$ set $\sigma_i(X_j) = b_{ij}X_j$. Then for each i , σ_i defines an automorphism of \mathcal{A}_{i-1} .

Then it satisfies (EA).

Proof: As in ([DCP93]) it follows that the algebra is an iterated twisted polynomial algebra. Suppose that (EA) is not satisfied. Let $u \in I_R$ be the smallest element which does not satisfy (EA). Then up to this order, the algebra behaves exactly as an iterated twisted polynomial algebra. But the advent of u then implies that there is at least one extra relation at this level. But this contradicts the fact that the algebra has the same Hilbert series as its associated quasipolynomial algebra (the algebra where the relations are $X_iX_j = b_{ij}X_jX_i$). \square

Remark 4.5. *It would be interesting to classify all quadratic algebras that satisfy this reduction assumption (EA) or, equivalently, (DCP). It is clearly a quite strong assumption, on the order of complication of e.g. the Jacobi Identity in the enveloping algebra.*

In [Ber78, Theorem 1.2], Bergman goes on to define a product and projection etc. but we are after something else - though also a projection.

5. THE CONSTRUCTION

Maintain the notation of Section 4.

Definition 5.1. *We define a linear map $S : V \otimes V \longrightarrow V \otimes V$ by*

$$S(X_i \otimes X_j) = b_{ij}X_jX_i + p_{ij} \text{ if } i > j, \quad (25)$$

$$S(X_j \otimes X_i) = (b_{ij})^{-1}(X_iX_j - p_{ij}) \text{ if } i > j, \text{ and} \quad (26)$$

$$S(X_i \otimes X_i) = X_i \otimes X_i \text{ for all } i = 1, \dots, N. \quad (27)$$

Furthermore, we define $\overline{S} : V \otimes V \longrightarrow V \otimes V$ by

$$\overline{S}(X_i \otimes X_j) = b_{ij}X_jX_i \text{ if } i > j, \quad (28)$$

$$\overline{S}(X_j \otimes X_i) = (b_{ij})^{-1}(X_iX_j) \text{ if } i > j, \text{ and} \quad (29)$$

$$\overline{S}(X_i \otimes X_i) = X_i \otimes X_i \text{ for all } i = 1, \dots, N. \quad (30)$$

From now on, we assume that

$$\forall i, j : b_{ij} = q^{a_{ij}}, \quad (31)$$

where q until further notice is a non-zero complex number. Recall that the associated quasi-polynomial algebra is the quadratic algebra $\overline{\mathcal{A}}$, generated (for clarity) by elements x_1, \dots, x_N with relations $x_i x_j = q^{a_{ij}} x_j x_i$.

Definition 5.2. For $i \in \mathbb{N}$, σ_i denotes the linear map $T \longrightarrow T$ given by

$$\sigma_i(v_1 \otimes \dots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_n) \quad (32)$$

$$= v_1 \otimes \dots \otimes v_{i-1} \otimes S(v_i \otimes v_{i+1}) \otimes \dots \otimes v_n \quad (33)$$

and $\overline{\sigma}_i$ denotes the linear map $T \longrightarrow T$ given by

$$\overline{\sigma}_i(v_1 \otimes \dots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_n) \quad (34)$$

$$= v_1 \otimes \dots \otimes v_{i-1} \otimes \overline{S}(v_i \otimes v_{i+1}) \otimes \dots \otimes v_n. \quad (35)$$

We now state and prove a series of lemmas about these maps.

Lemma 5.3. For each i ,

$$u_1 \leq_l u_2 \Leftrightarrow \overline{\sigma}_i(u_1) \leq_l \overline{\sigma}_i(u_2) \Leftrightarrow \sigma_i(u_1) \leq_l \sigma_i(u_2). \quad (36)$$

Proof: Clear from the definitions. \square

Lemma 5.4. For each $i \in \mathbb{N}$, σ_i is equal to the identity modulo I_R , i.e. for each $u \in T$ there exists an $r \in I_R$ such that

$$\sigma_i(u) = u + r. \quad (37)$$

Proof: This is obvious from the definitions. \square

Lemma 5.5. For each $i \in \mathbb{N}$, $\overline{\sigma}_i \overline{\sigma}_{i+1} \overline{\sigma}_i = \overline{\sigma}_{i+1} \overline{\sigma}_i \overline{\sigma}_{i+1}$, and hence $\overline{\sigma}_1, \dots, \overline{\sigma}_{n-1}$ define a representation, called **quasi-permutation**, of the symmetric group S_n on T^n .

Proof: By choosing a_{ij} appropriately, we may write $\overline{S}(X_i \otimes X_j) = q^{a_{ij}} X_j \otimes X_i$ for all $i, j = 1, \dots, N$. The claim follows easily from this by an elementary computation. \square

From now, in all statements involving *order*, we mean the lexicographical order \leq_l .

Lemma 5.6. For each $i \in \mathbb{N}$, $\sigma_i = \overline{\sigma}_i$ modulo lower order.

Proof: Obvious from the definitions. \square

Lemma 5.7. The following hold

1. For each $i \in \mathbb{N}$, σ_i preserves I_R .
2. For each $i \in \mathbb{N}$, if for $u \in T : \overline{\sigma}_i(u) = u$, then $\sigma_i(u) = u$.
3. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ modulo I_R or modulo lower order terms.

Proof: The first claim follows from Lemma 5.4. To prove the second claim it is clearly enough to prove that if for $u \in V \otimes V$, $\overline{S}(u) = u$, then $S(u) = u$, and for this, we may assume that

$$u = X_i \otimes X_j + q^{a_{ij}} X_j \otimes X_i. \quad (38)$$

The assertion then follows by an easy computation. The validity of the part of the last statement that involves I_R follows from Lemma 5.4 combined with the first item of this lemma. The validity of the other part follows from Lemma 5.6 combined with Lemma 5.5. \square

Lemma 5.8. *Let $u = a \cdot (X_i X_k - b_{ik} X_k X_i - p_{ik}) \cdot b \in I_R$, where $a, b \in \mathcal{A}$ and a is a homogeneous polynomial of degree $j - 1$. Then there exists a positive integer p such that $(1 + \sigma_j)^p u = 0$.*

Proof: We have that

$$(1 + \sigma_j) (a \cdot (X_i X_k - b_{ik} X_k X_i - p_{ik}) \cdot b) = a \cdot (1 - \sigma_j) p_{ik} \cdot b. \quad (39)$$

Since clearly, by construction and by Lemma 5.7, $(1 - \sigma_j) p_{ik}$ is of lower order, is in I_R , and is of the right form ((EA) is not needed here) one may repeat the procedure with u replaced by $u' = a \cdot (1 - \sigma_j) p_{ik} \cdot b$. After a finite number of steps one will reach 0. \square

We now wish to introduce an analogue of the usual symmetrization map on T . Let us first consider the representation of S_n described in Lemma 5.5. For any $\sigma \in S_n$ we denote the resulting operator on T^n as $\bar{\sigma}$ and we set

$$P_{\text{quasi-sym}} = \frac{1}{n!} \sum_{\sigma \in S_n} \bar{\sigma}, \quad (40)$$

and call this operator **quasi-symmetrization**. It is clear that this operator is the projection onto the subspace of tensors in T^n that are invariant under each $\bar{\sigma}_i, i = 1, \dots, n - 1$. More precisely, the following identities of course hold just as for ordinary symmetrization:

Lemma 5.9.

$$\forall i : \quad \bar{\sigma}_i \cdot P_{\text{quasi-sym}} = P_{\text{quasi-sym}} \cdot \bar{\sigma}_i = P_{\text{quasi-sym}}.$$

We next want to define a similar operator on T^n with respect to the σ_i 's. The problem is, of course, that we do not have a bona fide representation. In spite of this we proceed by defining for each $\sigma \in S_n$ an operator $\hat{\sigma} = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_r}$ if $\sigma = s_{i_1} s_{i_2} \cdots s_{i_r}$, where $s_j, j = 1, \dots, n - 1$, denotes the elementary transpositions in S_n and we set, for each such set of decompositions of elements,

$$P = \frac{1}{n!} \sum_{\sigma \in S_n} \hat{\sigma}. \quad (41)$$

Notice that for each $i = 1, \dots, n - 1$ we have a left coset decomposition of S_n with respect to the subgroup $\{1, s_i\}$; $S_n = C_i \times \{1, s_i\}$ for some suitable subset C_i of S_n . Hence we have, among the operators P , some of the form (all denoted P_i)

$$P_i = \tilde{P} \cdot (1 + \sigma_i). \quad (42)$$

More generally we can introduce

$$P_{i,r} = \tilde{P} \cdot \left(\frac{1 + \sigma_i}{2} \right)^r \quad (43)$$

where r later will be taken to be a sufficiently big power.

Corollary 5.10. *Each P leaves I_R invariant and $P = P_{\text{quasi-sym}}$ modulo lower order.*

Proof: This follows directly from Lemma 5.6 and Lemma 5.7. \square

Lemma 5.11. *Let $u \in I_R$. Then there exists an $N \in \mathbb{N}$ such that*

$$P^N(u) = 0. \quad (44)$$

Proof: By linearity and by Lemma 5.8, we may assume that $P_{i,r}(u) = 0$ for some i, r . But then, since P and P_i agree modulo lower order, $P(u)$ is of lower order than u . And by Corollary 5.10 $P(u) \in I_R$. Now invoke (EA) to yield that we after finitely many steps get $P^N(u) = 0$. \square

Corollary 5.12. *If $u \in I_R$ satisfies $P_{\text{quasi-sym}}(u) = u$ then $u = 0$.*

Proof: Combine Lemma 5.7 (2.) with Lemma 5.11. \square

Lemma 5.13. *Let $u \in T$. Then there exists an $N_0 \in \mathbb{N}$ such that*

$$P^N(u) = \tilde{P}^N(u) \quad (45)$$

for all $N \geq N_0$.

Proof: We have that $P(u) = P_{\text{quasi-sym}}(u) + u_1$ where u_1 is of lower order. By Lemma 5.6 and Lemma 5.7 it follows that $P^2(u) = P_{q\text{-sym}}(u) + P_{\text{quasi-sym}}(u_1) + u_2$. Thus, there exists a \hat{u} such that $P^N(u) = P_{\text{quasi-sym}}(\hat{u})$. Likewise, there exists a \tilde{u} such that $\tilde{P}^N(u) = P_{\text{quasi-sym}}(\tilde{u})$. Moreover, clearly

$$P_{\text{quasi-sym}}(\tilde{u}) = P_{\text{quasi-sym}}(\hat{u}) \mod I_R, \quad (46)$$

and hence, by Corollary 5.12 the claim follows. \square

Definition 5.14. *Set*

$$\mathcal{P}_{q\text{-sym}} = \lim_{N \rightarrow \infty} P^N. \quad (47)$$

The following is immediate

Proposition 5.15. *$\mathcal{P}_{q\text{-sym}}$ is a well-defined projection satisfying*

$$\mathcal{P}_{q\text{-sym}}(I_R) = 0. \quad (48)$$

Lemma 5.16. *If $P(u) = 0$, then $u \in I_R$. If $\mathcal{P}_{q\text{-sym}}(u) = 0$ then $u \in I_R$.*

Proof: This follows directly from Lemma 5.4. \square

Lemma 5.17. *If $\mathcal{P}_{q\text{-sym}}(u_1) = 0$ then $\mathcal{P}_{q\text{-sym}}(u_1 \otimes u) = 0$ for all $u \in T$.*

Proof: It follows by Lemma 5.16 that $u_1 \in I_R$. Hence $u_1 \otimes u \in I_R$. The claim then follows from Proposition 5.15. \square

We shall occasionally denote the restriction of $\mathcal{P}_{q\text{-sym}}$ to T^k by $\mathcal{P}_{q\text{-sym}}^k$, but most of the times we drop the subscript. For $r, s, k \in \mathbb{N}$ define the linear operator $I_r \otimes \mathcal{P}_{q\text{-sym}}^k \otimes I_s$ from T into T by

$$\begin{aligned} I_r \otimes \mathcal{P}_{q\text{-sym}}^k \otimes I_s(v_1 \otimes \cdots \otimes v_r \otimes v_{r+1} \otimes \cdots \otimes v_{r+k} \otimes v_{r+k+1} \otimes \cdots \otimes v_{r+k+s}) \\ v_1 \otimes \cdots \otimes v_r \otimes \mathcal{P}_{q\text{-sym}}^k(v_{r+1} \otimes \cdots \otimes v_{r+k}) \otimes v_{r+k+1} \otimes \cdots \otimes v_{r+k+s} \end{aligned} \quad (49)$$

The crucial property of $\mathcal{P}_{q\text{-sym}}$ then is

Proposition 5.18.

$$(\star) \quad \forall r, s : (I_r \otimes \mathcal{P}_{q\text{-sym}}^k \otimes I_s) \mathcal{P}_{q\text{-sym}}^{r+k-s} = \mathcal{P}_{q\text{-sym}}^{r+k-s}. \quad (50)$$

Proof: As in the proof of Lemma 5.13 observe that for any $u \in T$, $\mathcal{P}_{q\text{-sym}}(u)$ is quasi-symmetric. Hence the claim follows directly from Lemma 5.7 and Lemma 5.9. \square

We also have

Proposition 5.19.

$$(\star\star) \quad \forall r, s : \mathcal{P}_{q\text{-sym}}^{r+k-s}(I_r \otimes \mathcal{P}_{q\text{-sym}}^k \otimes I_s) = \mathcal{P}_{q\text{-sym}}^{r+k-s}. \quad (51)$$

Proof: By Proposition 5.15, it suffices to prove that for any $u \in T$, $(I_r \otimes \mathcal{P}_{q\text{-sym}}^k \otimes I_s)(u) - u \in I_R$. Here, it suffices to consider a u of the form $u_1 \otimes \cdots \otimes u_r \otimes v \otimes v_1 \otimes \cdots \otimes v_s$ with $v \in T$. Then

$$(I_r \otimes \mathcal{P}_{q\text{-sym}}^k \otimes I_s)(u) - u = u_1 \otimes \cdots \otimes u_r \otimes (\mathcal{P}_{q\text{-sym}}^k(v) - v) \otimes v_1 \otimes \cdots \otimes v_s \quad (52)$$

and the claim follows from Lemma 5.16 since by construction, I_R is an ideal in T . \square

Remark 5.20. *It is of course possible to introduce an inner product in $T(V)$ in which the projection $\mathcal{P}_{q\text{-sym}}$ is self-adjoint. Indeed, there is an infinite family of possible choices. It remains to be decided, if there is a natural candidate.*

6. DUALITY

6.1. New observations. We maintain the assumptions on \mathcal{A} . Let V^* denote the linear dual to V and denote the pairing by

$$V^* \times V \ni v^*, v \mapsto \langle v^*, v \rangle. \quad (53)$$

We extend this pairing to a pairing between $T^* = T(V^*)$ and T in the usual tensor product fashion.

Clearly, the introduced structure can be transported to T^* by this duality. On the level of the pairing between $V \otimes V$ and $V^* \otimes V^*$, we can consider the transposed of the S and \bar{S} of Definition 5.1. More generally, we can consider the projection $(\mathcal{P}_{q\text{-sym}})^t$ on T^* . Let I_R^t denote the kernel of the restriction of $(\mathcal{P}_{q\text{-sym}})^t$ to $V^* \otimes V^*$ and use I_R^t to define a quadratic algebra \mathcal{A}^t .

Proposition 6.1. \mathcal{A}^t is a quasipolynomial algebra.

Proof: This follows from condition (26) which implies that the columns in the matrix of $\mathcal{P}_{q\text{-sym}}$ corresponding to $X_i \otimes X_j$ and $b_{ij} \cdot X_j \otimes X_i$ have simple sums and differences. The transposed then have the same property for rows and this immediately gives that any pair X_i^*, X_j^* satisfies a quasipolynomial identity. Of course, there might a priori be more relations than that, but this is ruled out by dimension considerations in the dual algebra. \square

Remark 6.2. *Proposition 6.1 is perhaps surprising to the point of being disappointing. Notice however that the result relies on the chosen duality between $T(V)$ and $T(V^*)$. Other choices, e.g. based on inner products as in Remark 5.20 combined with a conjugation, may perhaps lead to other algebras, but we shall not pursue this point here.*

For $w \in T^{n*}$ and $z \in T^n$ we define the $\mathcal{P}_{q\text{-sym}}$ -symmetrized pairing $\langle\langle \cdot, \cdot \rangle\rangle$ by

$$\langle\langle w, z \rangle\rangle = n! \langle w, \mathcal{P}_{q\text{-sym}}(z) \rangle. \quad (54)$$

This is the pairing that generalizes the pairing $(q, p) = (q(\frac{\partial}{\partial z}), (p(\cdot))(0))$ between polynomials and differential operators.

Definition 6.3. For $w \in T^*$ and $z \in T$:

$$\boxed{\mathcal{F}_w(z) = \langle w, \mathcal{P}_{q\text{-sym}}(z) \rangle.} \quad (55)$$

It is clear that

$$\mathcal{F}_w(z) = \mathcal{F}_{[w]}([z]) \quad (56)$$

where $[z]$ and $[w]$ denote the equivalence classes in \mathcal{A} and \mathcal{A}^t , respectively, corresponding to z and w .

Definition 6.4.

$$\boxed{(\mathcal{F}_{[w_1]} \star \mathcal{F}_{[w_2]}) := \mathcal{F}_{[w_1 \otimes w_2]}.} \quad (57)$$

Proposition 6.5. The product in Definition 6.4 is a well-defined associative product.

Proof: The associativity is clear as soon as it is well-defined. This it is by (50). \square

Remark 6.6. Of course, there is the expected direct relation between the Poisson structure defined by the above non-commutative product,

$$\lim_{q \rightarrow 1} \frac{1}{q-1} (\mathcal{F}_{[w_1]} \star \mathcal{F}_{[w_2]} - \mathcal{F}_{[w_1]} \star \mathcal{F}_{[w_2]}),$$

and the usual Poisson structure for certain quadratic algebras as defined by Proceti and De Concini ([DCP93, p. 84-85]).

We consider ways of representing the functions $\mathcal{F}_{[w]}$ as functions on V .

Let X_1, \dots, X_N be a basis of V as in (Rel) in Section 4 and let

$$\forall \alpha = (\alpha_1, \dots, \alpha_N) \in (\mathbb{N}_0)^N : X^\alpha = X_1^{\alpha_1} \otimes \dots \otimes X_N^{\alpha_N}. \quad (58)$$

Furthermore, choose for each $\alpha \in (\mathbb{N}_0)^N$ a homogeneous polynomial

$$p_\alpha^{\mathcal{C}} = c_\alpha z_1^{\alpha_1} \dots z_N^{\alpha_N} + \sum_{\beta < \alpha} d_{\alpha, \beta} z^\beta$$

where each $d_{\alpha, \beta}$ is a complex number, where each c_α is a non-zero constant, and where the symbol \mathcal{C} (e.g. a lower triangular $\infty \times \infty$ matrix) represent these choices. The ordering $\beta < \alpha$ is lexicographic.

Definition 6.7.

$$\boxed{\mathcal{F}_{[w]}^{\mathcal{C}}(z_1, \dots, z_N) := \langle w, \mathcal{P}_{q\text{-sym}}(\sum_{\alpha} p_\alpha^{\mathcal{C}} X^\alpha) \rangle.}$$

The following is immediate

Proposition 6.8. *For each \mathcal{C} we get a faithful representation of the algebra \mathcal{A}^t in an associative algebra of polynomial functions on \mathbb{C}^N .*

The family of algebras we have defined by means of \mathcal{C} includes algebras defined by other (PBW-like) bases since a change of basis will simply be equivalent to a change of \mathcal{C} . For some specific choice of \mathcal{C} 's, a given element may give rise to a differential operator of an especially simple form, c.f. Section 7 below.

Remark 6.9. *We shall only pursue certain specific versions of Definition 6.7 below, but we wish to mention here that one may in fact go even further and represent the abstract functions of Definition 6.3 as non-commutative functions with values in certain algebras. In doing this, the construction is related to some algebras occurring when q is an m th root of unity. Specifically, for $M(n, \mathbb{C})$, observe that*

$$X_{1,1}^{a_{1,1}} \cdots X_{n,n}^{a_{n,n}} X_{1,1}^{m \cdot b_{1,1}} \cdots X_{n,n}^{m \cdot b_{n,n}} \quad (59)$$

with $0 \leq a_{i,j} \leq m-1$ for all $1 \leq i, j \leq n$ form a basis of \mathcal{A} for each $m \in \mathbb{N}$. Suppose namely that we could write 0 as a non-trivial linear combination of these. The coefficient of the highest order term is then by definition non-zero. However, we can rewrite the basis elements with respect to the standard basis. Doing this, the highest order term remains unchanged. But then the coefficient must be zero since the other basis is indeed a basis. Thus the elements are linearly independent, and by considering degrees, they must be a spanning set.

We can then interpret a specific element $X_{1,1}^{c_{1,1}} \cdots X_{n,n}^{c_{n,n}} X_{1,1}^{m \cdot d_{1,1}} \cdots X_{n,n}^{m \cdot d_{n,n}}$ in (59) as corresponding to the polynomial $z_{1,1}^{d_{1,1}} \cdots z_{n,n}^{d_{n,n}} \otimes (X_{1,1}^{c_{1,1}} \cdots X_{n,n}^{c_{n,n}})$ with values in the space spanned by the elements $X_{1,1}^{a_{1,1}} \cdots X_{n,n}^{a_{n,n}}$ with $0 \leq a_{i,j} \leq m-1$ for all $1 \leq i, j \leq n$.

We now consider, for $M(n, \mathbb{C})$, some specific instances of Definition 6.7:

Definition 6.10. *If $\{z_{i,j}\}_{i,j=1}^n \in M(n, \mathbb{C})$ set $z = \sum_{i,j=1}^n z_{i,j} X_{i,j}$ and*

$$\begin{aligned} \mathcal{F}_{[w]}^{(1)}(z_{11}, \dots, z_{n,n}) &= \langle w, \mathcal{P}_{q\text{-sym}}(Z \otimes \cdots \otimes Z) \rangle \\ \mathcal{F}_{[w]}^{(2)}(z_{11}, \dots, z_{n,n}) &= \langle w, \mathcal{P}_{q\text{-sym}}(\sum_{\alpha} c_{\alpha} z^{\alpha} X^{\alpha}) \rangle, \end{aligned} \quad (60)$$

where

$$c_{\alpha} = \frac{(|\alpha|)!}{(\alpha_{1,1})! \cdots (\alpha_{n,n})!}. \quad (61)$$

Now, let $[w_{\beta}]$ be determined by

$$\mathcal{F}_{[w_{\beta}]}^{(2)}(z_{1,1}, \dots, z_{n,n}) = z^{\beta}, \quad (62)$$

i.e.

$$\langle (\mathcal{P}_{q\text{-sym}})^t(w_{\beta}), X^{\alpha} \rangle = (c_{\beta})^{-1} \delta_{\alpha,\beta}. \quad (63)$$

By duality we have

$$\boxed{\frac{\partial}{\partial X_0} \mathcal{F}_{[w_{\beta}]}^{(i)}(\cdot) = \frac{1}{(|\beta| - 1)!} \langle \langle w_{\beta}, X_0(\cdot) \rangle \rangle}. \quad (64)$$

Thus,

$$\left(\frac{\partial}{\partial X_0} \mathcal{F}_{[w_\beta]}^{(i)} \right) (Z) = \begin{cases} |\beta| \cdot \langle w_\beta, \mathcal{P}_{q\text{-sym}}(X_0 \otimes Z \otimes \cdots \otimes Z) \rangle & \text{for } i = 1 \\ |\beta| \cdot \langle w_\beta, \mathcal{P}_{q\text{-sym}}(X_0 \otimes (\sum_\alpha c_\alpha z^\alpha X^\alpha)) \rangle & \text{for } i = 2 \end{cases} \quad (65)$$

If our ordering of X is $X_{1,1}, X_{1,2}, \dots, X_{n,n}$ then we get in particular that

$$\frac{\partial}{\partial X_{1,1}} \mathcal{F}_{[w_\beta]}^{(2)}(Z) = |\beta| \frac{c_\alpha z^\alpha}{c_\beta} \delta_{\alpha-1,\beta} = \beta_{1,1} z_{1,1}^{\beta_{1,1}-1} z_{1,2}^{\beta_{1,2}} \cdot z_{n,n}^{\beta_{n,n}}. \quad (66)$$

Likewise,

$$\begin{aligned} \frac{\partial}{\partial X_{1,2}} \mathcal{F}_{[w_\beta]}^{(2)}(Z) &= |\beta| \frac{c_\alpha z^\alpha}{c_\beta} \delta_{\alpha-1,\beta} = q^{-\beta_{1,1}} \beta_{1,2} z_{1,1}^{\beta_{1,1}} z_{1,2}^{\beta_{1,2}-1} \cdot z_{n,n}^{\beta_{n,n}}, \\ \frac{\partial}{\partial X_{2,1}} \mathcal{F}_{[w_\beta]}^{(2)}(Z) &= |\beta| \frac{c_\alpha z^\alpha}{c_\beta} \delta_{\alpha-1,\beta} = q^{-\beta_{1,1}} \beta_{2,1} z_{1,1}^{\beta_{1,1}} z_{1,2}^{\beta_{1,2}} z_{2,1}^{\beta_{2,1}-1} \cdot z_{n,n}^{\beta_{n,n}}, \end{aligned} \quad (67)$$

and (for 2×2) case

$$\begin{aligned} \frac{\partial}{\partial X_{2,2}} \mathcal{F}_{[w_\beta]}^{(2)}(Z) &= \beta_{2,2} q^{(-\beta_2-\beta_3)} z_{1,1}^{\beta_{1,1}} z_{1,2}^{\beta_{1,2}} z_{2,1}^{\beta_{2,1}} z_{2,2}^{\beta_{2,2}-1} \\ &\quad - q \cdot (1 - q^{-2\beta_{1,1}-2}) \frac{\beta_{1,2} \beta_{2,1}}{\beta_{1,1} + 1} z_{1,1}^{\beta_{1,1}+1} z_{1,2}^{\beta_{1,2}-1} z_{2,1}^{\beta_{2,1}-1} z_{2,2}^{\beta_{2,2}}. \end{aligned}$$

7. $M_q(2)$

We continue with the functions $\mathcal{F}_{[w_\beta]}^{(2)}(z_{1,1}, \dots, z_{n,n})$ from the previous section but specialize further to the quantized function algebra of 2×2 matrices.

Let $z_1 = z_{1,1}, z_2 = z_{1,2}, z_3 = z_{2,1}$, and $z_4 = z_{2,2}$. Then

$$\begin{aligned} \left(\frac{\partial}{\partial z_1} \right)_q (z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4}) &= \alpha_1 z_1^{\alpha_1-1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} \\ \left(\frac{\partial}{\partial z_2} \right)_q (z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4}) &= q^{-\alpha_1} \alpha_2 z_1^{\alpha_1} z_2^{\alpha_2-1} z_3^{\alpha_3} z_4^{\alpha_4} \\ \left(\frac{\partial}{\partial z_3} \right)_q (z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4}) &= q^{-\alpha_1} \alpha_3 z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3-1} z_4^{\alpha_4} \\ \left(\frac{\partial}{\partial z_4} \right)_q (z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4}) &= q^{-\alpha_2-\alpha_3} \alpha_4 z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4-1} \\ &\quad + \alpha_2 \alpha_3 \mathcal{K}_{\alpha_1} q^{-2\alpha_1+2} z_1^{\alpha_1} z_2^{\alpha_2-1} z_3^{\alpha_3-1} z_4^{\alpha_4} \end{aligned}$$

where, with $q = e^\hbar$,

$$\begin{aligned} \mathcal{K}_{\alpha_1} &= -q^{2\alpha_1-1} (1 - q^{-2\alpha_1-2}) \frac{z_1}{\alpha_1 + 1} \\ &= -e^{-3\hbar} (2\hbar + \dots + \frac{(2\hbar)^n}{n!} (\alpha + 1)^{n-1} + \dots) \cdot z_1. \end{aligned} \quad (68)$$

If we let $S_1 = z_1 \frac{\partial}{\partial z_1}$ then we see that $\mathcal{K}_{\alpha_1} = \mathcal{K}_1$ independently of α_1 where the operator

$$\mathcal{K}_1 \equiv -e^{-3\hbar}(2\hbar + \dots + \frac{(2\hbar)^n}{n!}(S_1)^{n-1} + \dots) \cdot z_1 \quad (69)$$

only involves the variable z_1 . The factors $q^{-\alpha_1}$ and $q^{-\alpha_2-\alpha_3}$ can of course also be dealt with analogously. However, if we define

$$K_i(z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4}) = q^{-\alpha_i} z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} \quad i = 1, 2, 3, 4 \quad (70)$$

then these are just like the usual K operators and we may then also write (if there is no subscript on a differential operator it means that it is a classical differential operator)

$$\begin{aligned} \left(\frac{\partial}{\partial z_1}\right)_q &= \left(\frac{\partial}{\partial z_1}\right) \\ \left(\frac{\partial}{\partial z_2}\right)_q &= K_1 \left(\frac{\partial}{\partial z_2}\right) \\ \left(\frac{\partial}{\partial z_3}\right)_q &= K_1 \left(\frac{\partial}{\partial z_3}\right) \\ \left(\frac{\partial}{\partial z_4}\right)_q &= K_2 K_3 \left(\frac{\partial}{\partial z_4}\right) + \mathcal{K}_1 \left(\frac{\partial}{\partial z_2}\right)_q \left(\frac{\partial}{\partial z_3}\right)_q \\ \left(\frac{\partial}{\partial z_4}\right)_q &= K_2 K_3 \left(\frac{\partial}{\partial z_4}\right) + \mathcal{O}_1 \left(\frac{\partial}{\partial z_2}\right) \left(\frac{\partial}{\partial z_3}\right), \end{aligned} \quad (71)$$

where $\mathcal{O}_1 = \mathcal{K}_1 K_1^2$.

Notice that $e^{-\hbar S} = K_1$, $\frac{\partial}{\partial z_1} \cdot \mathcal{K}_1 = -e^{-3\hbar}(e^{\hbar(2S+2)} - 1) = (q^{-3} - q^{-1}K_1^{-2})$, and $\frac{\partial}{\partial z_1} \cdot \mathcal{O}_1 = (q^{-3}K_1^2 - q^{-1})$.

The operators $\left(\frac{\partial}{\partial z_i}\right)_q, i = 1, 2, 3, 4$, satisfy similar relations as (14) for $X_{1,1}, X_{2,1}, X_{1,2}, X_{2,2}$ except that $q \rightarrow q^{-1}$. In particular, what corresponds to the wave operator \square_q is the central element

$$\square_q = \left(\frac{\partial}{\partial z_1}\right)_q \left(\frac{\partial}{\partial z_4}\right)_q - q^{-1} \left(\frac{\partial}{\partial z_2}\right)_q \left(\frac{\partial}{\partial z_3}\right)_q. \quad (72)$$

It is perhaps somewhat surprising that in this case the mixed degrees disappear again and

$$\square_q = K_2 K_3 \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_4} - q \frac{\partial}{\partial z_2} \frac{\partial}{\partial z_3}. \quad (73)$$

However, in the case of e.g. the Dirac operator, which basically will be a 2×2 matrix with entries (up to constant multiples) $\left(\frac{\partial}{\partial z_1}\right)_q, \dots, \left(\frac{\partial}{\partial z_4}\right)_q$, there is no cancelation of the second order term arising from $\left(\frac{\partial}{\partial z_4}\right)_q$.

We now discuss further the first order differential operators of (71). First of all we remark that the simple appearance of $\left(\frac{\partial}{\partial z_1}\right)_q$ is a result of the given choice of ordering. Other choices of orderings (or, equivalently, of constants c_α) can make the other variables have a simple appearance - at the expense of that of z_1 .

Secondly introduce the coordinate functions

$$\mathcal{F}_i^{(2)}(z_1, \dots, z_4) = z_i \text{ for } i = 1, 2, 3, 4. \quad (74)$$

We can then introduce the left derivatives δ_i^L for $i = 1, 2, 3, 4$:

$$\begin{aligned} & \left(\delta_i^L \mathcal{F}_{[w]}^{(2)} \right) (z_1, \dots, z_4) = \\ & \lim_{u \rightarrow 0} \left(\left(\mathcal{F}_i^{(2)}(ue_i) \right)^{-1} \star \left(\mathcal{F}_{[w]}^{(2)}((z_1, \dots, z_4) + ue_i) - \mathcal{F}_{[w]}^{(2)}(z_1, \dots, z_4) \right) \right) \end{aligned} \quad (75)$$

It follows easily that $\left(\frac{\partial}{\partial z_i} \right)_q = \delta_i^L$ for $i = 1, 2, 3$ and $\left(\frac{\partial}{\partial z_4} \right)_q = \delta_4^L + \mathcal{K}_1 \delta_2^L \delta_3^L$.

We finish this section with a study of how in particular $\left(\frac{\partial}{\partial z_4} \right)_q$ may be viewed as a covariant derivative. Let $\hat{\mathcal{O}} = \mathcal{O}_1 \left(\frac{\partial}{\partial z_2} \right) \left(\frac{\partial}{\partial z_3} \right)$. Set

$$F(f) = \begin{pmatrix} f \\ \hat{\mathcal{O}} f \\ (\hat{\mathcal{O}}^2 + [\hat{\mathcal{O}}, K_2 K_3 \left(\frac{\partial}{\partial z_4} \right)]) f \\ (\hat{\mathcal{O}}^3 + [\hat{\mathcal{O}}^2, K_2 K_3 \left(\frac{\partial}{\partial z_4} \right)] + K_2 K_3 \left(\frac{\partial}{\partial z_4} \right) [\hat{\mathcal{O}}, K_2 K_3 \left(\frac{\partial}{\partial z_4} \right)]) f \\ \vdots \end{pmatrix} \quad (76)$$

Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad (77)$$

Then

$$(K_2 K_3 \left(\frac{\partial}{\partial z_4} \right) + A) F(f) = F \left(\left(\frac{\partial}{\partial z_4} \right)_q f \right). \quad (78)$$

Another possibility is to let

$$G(f) = \begin{pmatrix} f \\ K_4^2 \left(\frac{\partial}{\partial z_2} \frac{\partial}{\partial z_3} \right) f \\ K_4^4 \left(\frac{\partial}{\partial z_2} \frac{\partial}{\partial z_3} \right)^2 f \\ K_4^6 \left(\frac{\partial}{\partial z_2} \frac{\partial}{\partial z_3} \right)^3 f \\ \vdots \end{pmatrix} \quad (79)$$

and

$$B = \begin{pmatrix} 0 & \mathcal{O}_1 K_4^{-2} & 0 & 0 & 0 & \dots \\ 0 & 0 & \mathcal{O}_1 K_4^{-2} & 0 & 0 & \dots \\ 0 & 0 & 0 & \mathcal{O}_1 K_4^{-2} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \quad (80)$$

Then we also have

$$(K_2 K_3 \left(\frac{\partial}{\partial z_4} \right) + B) G(f) = G \left(\left(\frac{\partial}{\partial z_4} \right)_q f \right). \quad (81)$$

This last version is also well behaved with respect to the other generators.

APPENDIX

Here we compute the first order differential operators for $M_q(n)$. Observe that we have

$$\begin{aligned} z_{n,i} (z_{a,1}^{\alpha_{a,1}} z_{a,2}^{\alpha_{a,2}} \cdots z_{a,n}^{\alpha_{a,n}}) = \\ \sum_{x=1}^{i-1} c_{a,x} q^{(\alpha_{a,x+1} + \cdots + \alpha_{a,i-1})} z_{a,1}^{\alpha_{a,1}} \cdots z_{a,x}^{\alpha_{a,x}-1} \cdots z_{a,i}^{\alpha_{a,i}+1} \cdots z_{a,n}^{\alpha_{a,n}} z_{n,x} \\ + q^{\alpha_{a,i}} (z_{a,1}^{\alpha_{a,1}} z_{a,2}^{\alpha_{a,2}} \cdots z_{a,n}^{\alpha_{a,n}}) z_{n,i}, \end{aligned} \quad (82)$$

where $c_{a,x} = q(q^{-2\alpha_{a,x}} - 1)$ and where the exponent to q should be interpreted as 0 for $x = i-1$.

With (82) to our disposal we can now give the general form of $\frac{\partial}{\partial X_{i,j}}$. Let $\Gamma_{i,j} = \Gamma_{i,j}^d \cup \Gamma_{i,j}^u$ denote the union of the following sets of “paths” from $(1, j)$ to $(i, 1)$:

$$\begin{aligned} \Gamma_{i,j}^d &= \{[i_1, \dots, i_r; j_1, \dots, j_r] \mid r \in \mathbb{N}, \\ 1 &= i_1 < i_2 < \cdots < i_r = i, 1 \leq j_r < \cdots < j_1 = j\}, \\ \Gamma_{i,j}^u &= \{[i_1, \dots, i_r; j_1, \dots, j_r] \mid r \in \mathbb{N}, \\ 1 &< i_1 < i_2 < \cdots < i_r = i, 1 \leq j_r < \cdots < j_1 = j\}. \end{aligned} \quad (83)$$

For $j = 1, i \geq 1$ we interpret the above as $\Gamma_{i,1}^d = \emptyset$ and $\Gamma_{i,1}^u = \{[i; 1]\}$ whereas for $i = 1, j > 1$ it is $\Gamma_{1,j}^u = \emptyset$ and $\Gamma_{1,j}^d = \{[1; j]\}$.

For each $z_{i,j}$ we define an operator $\mathcal{K}_{i,j}$ in analogy with (69) and an operator $K_{i,j}$ in analogy with (70), and finally we set $\mathcal{O}_{i,j} = \mathcal{K}_{i,j} K_{i,j}^2$.

For $g = [1_1, \dots, i_r; j_1, \dots, j_r] \in \Gamma_{i,j}^d$, set

$$\begin{aligned} S_g &= \{(s, t) \mid \exists x = 1, \dots, r-1 : s = i_x \text{ and } j_{x+1} < t < j_x\}, \\ T_g &= \{(s, t) \mid \exists x = 1, \dots, r-1 : t = j_{x+1} \text{ and } i_x < t < i_{x+1}\}, \\ \text{and} \end{aligned} \quad (84)$$

$$D_{i,j}^d(g) = \left(\prod_{y=1}^{r-1} \mathcal{O}_{i_y, j_{y+1}} \prod_{(s,t) \in S_g} K_{s,t} \prod_{(s,t) \in T_g} K_{s,t}^{-1} \prod_{x < j_r} K_{i,x}^{-1} \right) \prod_{x=1}^r \frac{\partial}{\partial z_{i_x, j_x}}.$$

Likewise, for $g = [1_1, \dots, i_r; j_1, \dots, j_r] \in \Gamma_{i,j}^u$, set, for convenience, $i_0 = 0$ and

$$\begin{aligned} U_g &= \{(s, t) \mid \exists x = 1, \dots, r-1 : s = i_{x+1} \text{ and } j_{x+1} < t < j_x\}, \\ V_g &= \{(s, t) \mid \exists x = 1, \dots, r : t = j_x \text{ and } i_{x-1} < t < i_x\}, \\ \text{and} \end{aligned} \quad (85)$$

$$D_{i,j}^u(g) = \left(\prod_{y=1}^{r-1} \mathcal{O}_{i_y, j_{y+1}} \prod_{(s,t) \in U_g} K_{s,t} \prod_{(s,t) \in V_g} K_{s,t}^{-1} \prod_{x < j_r} K_{i,x}^{-1} \right) \prod_{x=1}^r \frac{\partial}{\partial z_{i_x, j_x}}.$$

Then

$$\frac{\partial}{\partial X_{i,j}} = \sum_{g \in \Gamma_{i,j}^d} D_{i,j}^d(g) + \sum_{g \in \Gamma_{i,j}^u} D_{i,j}^u(g). \quad (86)$$

Observe that the lowest order differential operator occurring as a summand in $\frac{\partial}{\partial X_{i,j}}$ is $\left(\prod_{y < j_r} K_{1,y}^{-1} \prod_{x < i} K_{x,j}^{-1}\right) \frac{\partial}{\partial z_{i,j}}$.

It is not clear if an analogue of the operator G exists for higher order algebras.

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